$$w = u + j[1 - F(\theta', k_0')/K(k_0')] \quad (87)$$

$$k_0' = \sqrt{1 - k_0^2} \quad (88)$$

$$\sin \theta' = (1/k_0')\sqrt{1 - 1/t^2}. \quad (89)$$

Eqs. (87), (89), (83), and (81) give the implicit relation between x and v along the matching surfaces.

Fig. 11 shows the z plane for the even mode operation of the coupled strip line. From it, it can be seen that the t plane, the t' plane, and the w plane sketches are identical with the odd mode sketches, except for replacing k_0 by k_e. Eqs. (89) and (86) apply, and by analogy with (85),

$$w = [K(k_e)/K(k_e')][1 - F(\theta, k_e)/K(k_e)] + j1. \quad (90)$$

Along $y=0$, by analogy with (87),

$$w = u + j[1 - F(\theta', k_e')/K(k_e')]. \quad (91)$$

Eq. (79) applies to Fig. 11 as well as to Fig. 10. When it is integrated, and boundary conditions are applied, the result is

$$z = (a/\pi) \ln [\sqrt{t'/p} + \sqrt{(t'/p) - 1}]. \quad (92)$$

Along $y=0$ this can be written as

$$t' = \frac{1 - k_e}{1 + k_e} \frac{\cosh^2 \pi x/a}{\cosh^2 \pi s/2a}. \quad (93)$$

By considering the value of t' at $x=s/2$ and at $x=(w+s/2)$ the expression for k_e can be obtained:

$$k_e = (\tanh \pi w/2a)(\tanh \pi(w + s)/2a). \quad (94)$$

Eqs. (91), (89), (83), and (93) serve to give the implicit relation between x and v along $y=0$.

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The Impedance of a Wire Grid Parallel to a Dielectric Interface*

JAMES R. WAIT†

Summary—**Analysis is given for the problem of reflection of a plane wave at oblique incidence on a wire grid which is parallel to a plane interface between two homogeneous dielectrics. It is assumed that the wire grid is a periodic structure and consists of thin cylindrical wires of homogeneous material. The equivalent circuit is derived where it is shown that the space on either side of the interface can be represented by a transmission line, and the grid itself is represented by a pure shunt element across one of the lines.**

Introduction

There have been many investigations of the electromagnetic properties of thin parallel wires composed of conductive material. The first quantitative study was made by Lamb[1] in 1898 who considered the plane wave incident normally on the grid. He showed that if the diameter, $2a$, of the parallel wires was small, the reflection and transmission could be varied by changing the spacing. In 1914, von Ignatowsky[2] made a very exhaustive analysis of the scattering of incident plane waves by single metallic grids including the case where the wire spacing is comparable to the wavelength. His formulas have been reduced, extended, and applied by other authors since that time.[3–11] A very illuminating treatment has been given by MacFarlane[5] who indicated that a single grid can be represented by an impedance shunted across an infinite transmission line whose characteristic impedance is proportional to the intrinsic impedance of the

* Manuscript received by the PGMTT, June 1, 1956.
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Note: In this reference $\cos \theta \cos \phi_0 Z_i d/\eta_0$ should be replaced by $\cos^{-1} \theta \cos \phi_0 Z_i d/\eta_0$ in (25).

surrounding infinite medium. He showed that this shunt impedance was proportional to log $(d/2\pi a)+F(\theta, d)$ where F is a correction factor which is a function of angle of incidence θ and the spacing d.

Recently it has been suggested by Jones and Cohn[12] that a wire grid, if suitably located near an air-interface of a dielectric lens, can effectively simulate a quarter-wave matching network. Considering the case where the wire grid is embedded within the lens medium of intrinsic impedance Z, it can be readily shown[12] that the conditions for the matching the lens to the exterior (air) medium of intrinsic impedance Z' are

$$X_g/Z = \tan 2\phi$$

and

$$Z'/Z = \tan^2 \phi$$

where ϕ is the "electrical" distance in radians from the lens surface to the embedded grid whose impedance is $Z_g \cong iX_g$.

The use of MacFarlane's expression for Z_g in the above application is not strictly correct as the $F(\theta, d)$ function that he rigorously computed is only valid in the case where the grid is in an infinite medium. For dielectric lens matching, the value of ϕ is usually of the order of $\pi/4$ radians corresponding to an air-lens interface to grid separation of the order of $\frac{1}{8}$ wavelength. It will be shown that some error can be introduced if account is not taken of the interface. The following analysis treats fully the problem of a parallel wire grid embedded in a homogeneous dielectric at a fixed distance from a plane interface. While the analysis employs the terminology appropriate to lossless dielectrics, the extension to dissipative media on either or both sides of the interface is effected by regarding the respective dielectric constants to be complex.

Outline of Solution

With respect to a Cartesian coordinate system, the wire grid is contained in the plane $x=h$ and is parallel to a plane interface, at $x=0$, of two dielectrics. The grid is composed of an array of wires parallel to the z axis and spaced a distance d between centers. The wires are taken to be of circular cross section and the diameter is assumed to be small compared to d. The medium $(x>0)$ surrounding the wire grid is homogeneous with a dielectric constant ϵ. The homogeneous medium beyond the interface $(x<0)$ has a dielectric constant ϵ'.

A plane wave with the electric field of magnitude E_0 parallel to the z axis, impinges on the grid with angle of incidence θ as indicated in Fig. 1. The primary or incident field is given by

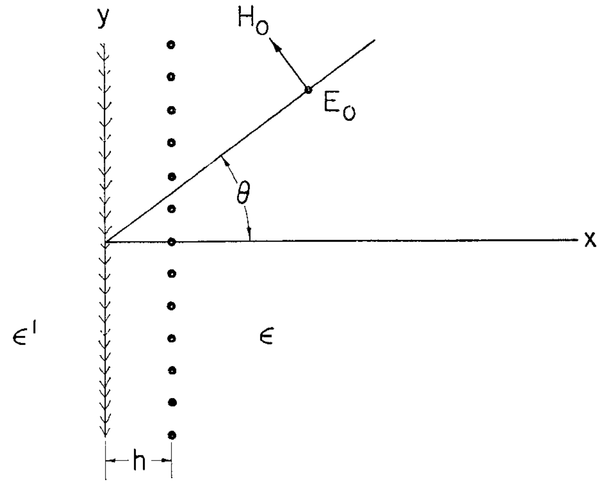

Fig. 1—The wire grid parallel to a plane interface between two dielectrics.

$$E_i = E_0 \exp\left[i2\pi\lambda^{-1}(x \cos\theta + y \sin\theta)\right] \tag{1}$$

where the time factor exp $(i\omega t)$ has been omitted and

$$2\pi\lambda^{-1} = (\epsilon\mu)^{1/2}\omega \tag{2}$$

where λ is the wavelength in the incident medium and μ is the permeability which is assumed to be the same for both dielectrics. The currents induced on the wires will all be of equal magnitude I but will have a progressive change of phase between adjacent wires of $2\pi\lambda^{-1}d \sin\theta$ radians.

In the absence of a grid a reflected wave

$$E_r = E_0 R \exp\cdot\left[i(2\pi/\lambda)(-x\cos\theta + y\sin\theta)\right] \tag{3}$$

for $x>0$, and a transmitted wave

$$E_t = E_0 T \exp\cdot\left[i(2\pi/\lambda')(x\cos\theta' + y\sin\theta')\right] \tag{4}$$

for $x<0$, will be set up. R and T are Fresnel reflection coefficients given by

$$R = T - 1 = (K' - K)/(K' + K) \tag{5}$$

where

$$K = \eta/\cos\theta, \quad K' = \eta'/\cos\theta', \quad \eta = (\mu/\epsilon)^{1/2}$$

and

$$\sin\theta' = (\lambda'/\lambda)\sin\theta = (\epsilon/\epsilon')^{1/2}\sin\theta = (\eta'/\eta)\sin\theta.$$

It is now necessary to consider the field E_w of the wire grid carrying a current of magnitude I. From a previous analysis,[10] it is known that:

$$E_w = \frac{i\mu\omega I}{4\pi}\exp\cdot\left[i(2\pi/\lambda)y\sin\theta\right]\cdot\sum_{m=-\infty}^{+\infty}\frac{\exp\left[i2\pi my/d\right]\exp\left[-(2\pi/d)\,|\,x-h\,|\sqrt{(m+D\sin\theta)^2-D^2}\right]}{\sqrt{(m+D\sin\theta)^2-D^2}} \tag{6}$$

where $D=d/\lambda$. To satisfy the boundary conditions at the interface, a further secondary field $E_w{}^s$ in the incident medium must be introduced, and further transmitted field $E_w{}^t$, given by

[12] E. M. T. Jones and S. B. Cohn, "Surface matching of dielectric lenses," *J. Appl. Phys.*, vol. 26, pp. 452–457; April, 1955.

$$E_w{}^s = \frac{i\mu\omega I}{4\pi} \exp\left[i(2\pi/\lambda)y\sin\theta\right]\cdot \sum_{m=-\infty}^{+\infty} \frac{R_m \exp\left[i2\pi my/d\right]\exp\left[-(2\pi/d)(x+h)\sqrt{(m+D\sin\theta)^2-D^2}\right]}{\sqrt{(m+D\sin\theta)^2-D^2}} \tag{7}$$

and

$$E_w{}^t = \frac{i\mu\omega I}{4\pi} \exp\left[i(2\pi/\lambda)y\sin\theta\right]\cdot \sum_{m=-\infty}^{+\infty} \frac{T_m \exp\left[i2\pi my/d\right]\exp\left[(2\pi/d)x\sqrt{(m+D\sin\theta)^2-(D')^2}\right]}{\sqrt{(m+D\sin\theta)^2-D^2}\exp\left[(2\pi/d)h\sqrt{(m+D\sin\theta)^2-D^2}\right]} \tag{8}$$

where $D'=d/\lambda'$. R_m and T_m are analogous to Fresnel reflection coefficients and are given by

$$R_m = T_m - 1 = \frac{\sqrt{(m+D\sin\theta)^2-D^2}-\sqrt{(m+D\sin\theta)^2-(D')^2}}{\sqrt{(m+D\sin\theta)^2-D^2}+\sqrt{(m+D\sin\theta)^2-(D')^2}}. \tag{9}$$

where

$$Z_g = \frac{i\mu\omega d}{2\pi}\left(\log\frac{d}{2\pi a}+\Delta\right)+(1+i)\,\frac{d}{2\pi a}\left(\frac{\bar{\mu}\omega}{2\bar{\sigma}}\right)^{1/2} \tag{12}$$

with

$$\Delta = \frac{1}{2}\sum_{m=1}^{\infty}\left[\frac{1+R_m\exp\left[-4\pi HD^{-1}\sqrt{(m+D\sin\theta)^2-D^2}\right]}{\sqrt{(m+D\sin\theta)^2-D^2}} + \frac{1+R_{-m}\exp\left[-4\pi HD^{-1}\sqrt{(m-D\sin\theta)^2-D^2}\right]}{\sqrt{(m-D\sin\theta)^2-D^2}} - \frac{2}{m}\right] \tag{13}$$

The total field is then

$$E = E_i + E_r + E_w + E_w{}^s \qquad \text{for } x > 0,$$
$$= E_t + E_w{}^t \qquad \text{for } x < 0.$$

It can readily be verified that E and $\partial E/\partial x$ are continuous at $x=0$.

The value of the current I is now found from the additional boundary condition that $E=-Iz_i \exp[i(2\pi/\lambda)y\sin\theta]$ where E is the field at the surface of the wire and z_i is the internal impedance of the wire. It can be assumed that the field is uniform around the wire since $a \ll d$ and $a \ll \lambda$ and hence z_i can be calculated by known methods and is given by[11]

$$z_i = \frac{\bar{\eta} I_0(\bar{\gamma}a)}{2\pi a I_1(\gamma a)} \tag{10a}$$

where $\bar{\eta}=[i\bar{\mu}\omega/(\bar{\sigma}+i\omega\bar{\epsilon})]^{1/2}$ and $\bar{\gamma}=[i\bar{\mu}\omega(\bar{\sigma}+i\omega\bar{\epsilon})]^{1/2}$ and $\bar{\mu}$, $\bar{\sigma}$, and $\bar{\epsilon}$ are the permeability, conductivity, and dielectric constant of the wire material. I_0 and I_1 are modified Bessel functions of or order of zero and unity. For metallic wires, the displacement currents are negligible since $\omega\bar{\epsilon} \ll \bar{\sigma}$ even for microwaves. In addition, the frequency is usually sufficiently high so that $|\bar{\gamma}a| \gg 1$ and hence

$$z_i \simeq \left(\frac{\bar{\mu}\omega}{2\sigma}\right)^{1/2}\frac{1+i}{2\pi a}. \tag{10b}$$

Invoking the boundary condition at the wire leads to

$$I = \frac{E_0 d\left\{\exp\left[i(2\pi/\lambda)h\cos\theta\right]+R\exp\left[-i(2\pi/\lambda)h\cos\theta\right]\right\}}{(\eta/2\cos\theta)(1+R\exp\left[-i(4\pi/\lambda)\,h\cos\theta\right])+Z_g} \tag{11}$$

where R_m is defined in (9) and R_{-m} is obtained by replacing m with $-m$. The current I on the grid wires is now specified in terms of known quantities. An immediate partial check is obtained by noting that I reduces to (10) of an earlier paper[10] when ϵ' approaches infinity corresponding to a perfectly conducting plane at $x=0$.

The Distant Field

Although the complete solution of the problem has now been obtained, it is very desirable to focus attention on the distant scattered field. For example, if $(x=h)\gg\lambda$ it is evident that only the terms for $m=0$ are significant for $d/\lambda<1/(1+\sin|\theta|)$ and $d/\lambda'<1/(1+\sin|\theta'|)$. The higher values of m correspond to evanescent waves which are highly damped in the positive and negative x directions. For larger values of d, additional undamped waves can be scattered from the grid. The discussion will be limited here to the smaller grid spacings satisfying the above inequality. The distant fields are then given by

$$E = E_0\exp\left[i(2\pi/\lambda)(x\cos\theta+y\sin\theta)\right] + \left\{E_0 R + \frac{I\eta}{2d\cos\theta}\left[\exp(i2\pi H\cos\theta) + R\exp(-i2\pi H\cos\theta)\right] \cdot\exp\left[i(2\pi/\lambda)\cdot(-x\cos\theta+y\sin\theta)\right]\right\} \tag{14}$$

for large positive x, and

$$E = \left\{E_0 T + \frac{I\eta}{2d\cos\theta} T \exp(-i2\pi H\cos\theta)\right\} \cdot \exp\left[i(2\pi/\lambda')(x\cos\theta' + y\sin\theta')\right] \qquad (15)$$

for large negative x, where

$$R = T - 1 = (K' - K)/(K' + K) \qquad (16)$$

with $K = \eta/\cos\theta$ and $K' = \eta'/\cos\theta'$ with $H = h/\lambda$.

The equivalent circuit, which may be taken as the analog of the wire grid is shown in Fig. 2. The space to the right of the interface, $(x > 0)$, is represented by a transmission line of characteristic impedance K and propagation constant Γ. The line constants for the space to the left $(x < 0)$ are K' and Γ'. At $x = h$, the line is *shunted* by an impedance Z_g. The voltage V across the line and the current in the line J can now be identified with the electric field E and the magnetic field component H_y respectively. The characteristic impedances have been defined earlier and the propagation constants are given by

$$\Gamma = i(2\pi/\lambda)\cos\theta \quad \text{and} \quad \Gamma' = i(2\pi/\lambda')\cos\theta'.$$

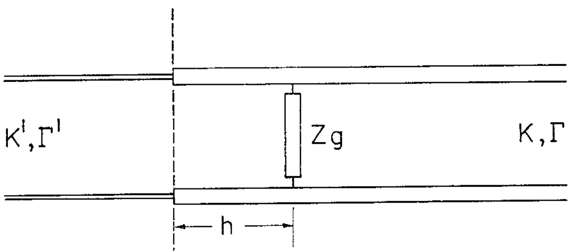

Fig. 2—The equivalent circuit consisting of two semi-infinite transmission lines with shunt element across one.

In the case of normal incidence $(\theta = \theta' = 0)$, (12) reduces to

$$Z_g = i\frac{d}{\lambda}\eta\left[\log\frac{d}{2\pi a} + \Delta\right] + (1 + i)\frac{d}{2\pi a}\left(\frac{\bar{\mu}\omega}{2\bar{\sigma}}\right) \qquad (17)$$

$$\Delta = \sum_{m=1}^{\infty}\left\{\frac{1 + R_m \exp\left[-4\pi(H/D)\sqrt{m^2 - D^2}\right]}{\sqrt{m^2 - D^2}} - \frac{1}{m}\right\} \qquad (18)$$

and

$$R_m = \frac{\sqrt{m^2 - D^2} - \sqrt{m^2 - (ND)^2}}{\sqrt{m^2 - D^2} + \sqrt{m^2 - (ND)^2}}. \qquad (19)$$

Numerical Discussion

Some numerical values of Δ for normal incidence $(\theta = 0)$ and $\lambda'/\lambda = 1.57$ are given in Table I. The ratio of the wavelengths between the medium surrounding the grid and the exterior medium (*i.e.*, 1.57) is typical for a dielectric lens. The case of $H = 0$ corresponds to the grid in the interface and $H = \infty$ corresponds to the grid within the dielectric lens medium when the interface is effectively at an infinite distance. Since for reactive-wall matching, H is of the order $\frac{1}{8}$, it is seen that the correction factor Δ can be appreciably different from the corresponding value for $H = \infty$ if D is of the order of one half-wave length or greater. When the grid is in the interface, the value of Δ can be expected to be influenced equally by the electrical properties of both media. This is demonstrated by considering the case $H = 0$, $\theta = \theta' = 0$, whence[13]

TABLE I
VALUES OF Δ

$D=$	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$H=0$	0.0170	0.0456	0.0728	0.1210	0.1865	0.1262	0.2216
$=\frac{1}{16}$	0.0245	0.0567	0.1027	0.1620	0.2561	0.3813	0.4614
$=\frac{1}{8}$	0.0246	0.0573	0.1067	0.1774	0.2771	0.4227	0.5856
$=\frac{1}{2}$	0.0246	0.0574	0.1068	0.1808	0.2882	0.4532	0.7433
$=\infty$	0.0246	0.0575	0.1068	0.1809	0.2882	0.4535	0.7477

$$\Delta = \sum_{m=1}^{\infty}\frac{2}{\sqrt{m^2 - D^2} + \sqrt{m^2 - (ND)^2}} - \frac{1}{m} \qquad (20)$$

which can be approximated by

$$\Delta \simeq 0.301(N^2 + 1)D^2 \qquad (21)$$

subject to $(ND)^4$ and $(D)^4 \ll 1$. This can be rewritten

$$\Delta \simeq 0.601(k_e D/2\pi)^2 \qquad (22)$$

where ik_e is the effective propagation constant for a thin wire in an interface between two media whose propagation constants are $ik(=2\pi i/\lambda')$ and $ik'(=2\pi i/\lambda')$. Therefore subject to the above approximations

$$ik_e = i\sqrt{[(k)^2 + (k')^2]/2}. \qquad (23)$$

This simple formula for the effective propagation constant was suggested previously on intuitive grounds.[14]

Conclusion

This analysis provides an exact expression for the equivalent shunt impedance Z_g of a thin wire grid situated parallel to the interface between two homogeneous dielectrics. The results indicate that Z_g is dependent on the distance h from the grid to the interface. The error incurred, however, in neglecting this effect in applications to surface-matching[12] of dielectric lenses is unimportant if the grid spacing d is less than about 0.5λ for normal incidence. This condition can become more stringent when the E vector is not parallel to the wires.[15]

[13] This particular formula is identical to one quoted to me by G. D. Monteath of the British Broadcasting Corp. recently. He says it can be obtained directly using a quasi-static method. I am indebted to Mr. Monteath for a stimulating discussion of this matter.

[14] J. R. Wait and W. A. Pope, "Input resistance of l. f. unipole aerials (with radial wire earth systems)," *Wireless Eng.*, vol. 32, pp. 131–138; May, 1955.

[15] J. R. Wait, "On the theory of reflection (at arbitrary incidence) from a wire grid parallel to an interface between homogeneous media," *Appl. Sci. Res.*, vol. BV, 1957 (in press).